

# Periodic Solutions for Circular Restricted 4-body Problems with Newtonian Potentials <sup>\*</sup>

Xiaoxiao Zhao, and Shiqing Zhang

Yangtze Center of Mathematics and College of Mathematics, Sichuan University,  
Chengdu 610064, People's Republic of China

**Abstract:** We study the existence of non-collision periodic solutions with Newtonian potentials for the following planar restricted 4-body problems: Assume that the given positive masses  $m_1, m_2, m_3$  in a Lagrange configuration move in circular orbits around their center of masses, the sufficiently small mass moves around some body. Using variational minimizing methods, we prove the existence of minimizers for the Lagrangian action on anti- $T/2$  symmetric loop spaces. Moreover, we prove the minimizers are non-collision periodic solutions with some fixed winding numbers.

**Keywords:** Restricted 4-body problem; non-collision periodic solution; variational minimizer; winding number.

2000 AMS Subject Classification 34C15, 34C25, 58F.

## 1 Introduction and Main Results

In this paper, we study the planar circular restricted 4-body problems with Newtonian potentials. Suppose points of positive masses  $m_1, m_2, m_3$  move in a plane of their circular orbits  $q_1(t), q_2(t), q_3(t)$  and the center of masses is at the origin; suppose the sufficiently small mass point does not influence the motion of  $m_1, m_2, m_3$ , and moves in the plane for the given masses  $m_1, m_2, m_3$ .

It is well-known that  $q_1(t), q_2(t), q_3(t)$  satisfy the Newtonian equations:

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i = 1, 2, 3, \quad (1.1)$$

where

$$U = \sum_{1 \leq i < j \leq 3} \frac{m_i m_j}{|q_i - q_j|}. \quad (1.2)$$

Without loss of generality, we assume that there exists  $\theta_1, \theta_2, \theta_3 \in [0, 2\pi)$  such that the planar circular orbits are

$$q_1(t) = r_1 e^{\sqrt{-1} \frac{2\pi}{T} t} e^{\sqrt{-1} \theta_1}, \quad q_2(t) = r_2 e^{\sqrt{-1} \frac{2\pi}{T} t} e^{\sqrt{-1} \theta_2}, \quad q_3(t) = r_3 e^{\sqrt{-1} \frac{2\pi}{T} t} e^{\sqrt{-1} \theta_3}, \quad (1.3)$$

where the radius  $r_1, r_2, r_3$  are positive constants depending on  $m_i (i = 1, 2, 3)$  and  $T$  (see Lemma 2.6). We also assume that

$$m_1 q_1(t) + m_2 q_2(t) + m_3 q_3(t) = 0 \quad (1.4)$$

and

$$|q_i - q_j| = l, \quad 1 \leq i \neq j \leq 3, \quad (1.5)$$

where the constant  $l > 0$  depends on  $m_i (i = 1, 2, 3)$  and  $T$  (see Lemma 2.5).

---

<sup>\*</sup>Supported by National Natural Science Foundation of China.

The orbit  $q(t) \in R^2$  for sufficiently small mass is governed by the gravitational forces of  $m_1, m_2, m_3$  and therefore it satisfies the following equation

$$\ddot{q} = \sum_{i=1}^3 \frac{m_i(q_i - q)}{|q_i - q|^3}. \quad (1.6)$$

For  $N$ -body problems, there are many papers concerned with the periodic solutions by using variational methods, see [1-9,13-16,18] and the references therein. In [3], Chenciner-Montgomery proved the existence of the remarkable figure-“8” type periodic solution for planar Newtonian 3-body problems with equal masses. Marchal[6] studied the fixed end problem for Newtonian  $n$ -body problems and proved the minimizer for the Lagrangian action has no interior collision. Especially, in [8], Simó used computer to discover many new periodic solutions for Newtonian  $n$ -body problems. Zhang-Zhou[13-15] decomposed the Lagrangian action for  $n$ -body problems into some sum for two body problems and [14,15] avoid collisions by comparing the lower bound for the Lagrangian action on the symmetry collision orbits and the upper bound for the Lagrangian action on test orbits in some cases.

Motivated by the above works, we use variational methods to study the circular restricted 3+1-body problem with some fixed wingding numbers and some masses.

For the readers' conveniences, we recall the definition of the winding number, which can be found in many books on the classical differential geometry.

**Definition 1.1** Let  $\Gamma : x(t), t \in [a, b]$  be an given oriented continuous closed curve, and  $p$  be a point of the plane not on the curve. Then, the mapping  $\varphi : \Gamma \rightarrow S^1$ , given by

$$\varphi(x(t)) = \frac{x(t) - p}{|x(t) - p|}, \quad t \in [a, b]$$

is defined to be the position mapping of the curve  $\Gamma$  relative to  $p$ , when the point on  $\Gamma$  goes around the curve once, its image point  $\varphi(x(t))$  will go around  $S^1$  a number of times, this number is called the winding number of the curve  $\Gamma$  relative to  $p$ , and we denote it by  $\deg(\Gamma, p)$ . If  $p$  is the origin, we write  $\deg\Gamma$ .

Define

$$W^{1,2}(R/TZ, R^2) = \left\{ x(t) \middle| x(t), \dot{x}(t) \in L^2(R, R^2), x(t+T) = x(t) \right\}.$$

The norm of  $W^{1,2}(R/TZ, R^2)$  is

$$\|x\| = \left[ \int_0^T |x|^2 dt \right]^{\frac{1}{2}} + \left[ \int_0^T |\dot{x}|^2 dt \right]^{\frac{1}{2}}. \quad (1.7)$$

The functional corresponding to the equation (1.6) is

$$f(q) = \int_0^T \left[ \frac{1}{2} |\dot{q}|^2 + \sum_{i=1}^3 \frac{m_i}{|q - q_i|} \right] dt, \quad q \in \Lambda_{\pm}, \quad (1.8)$$

where

$$\Lambda_- = \left\{ q \in W^{1,2}(R/TZ, R^2) \middle| \begin{array}{l} q(t + \frac{T}{2}) = -q(t), \deg(q - q_1) = -1, \\ q(t) \neq q_i(t), \forall t \in [0, T], i = 1, 2, 3 \end{array} \right\}.$$

and

$$\Lambda_+ = \left\{ q \in W^{1,2}(R/TZ, R^2) \middle| \begin{array}{l} q(t + \frac{T}{2}) = -q(t), \deg(q - q_1) = 1, \\ q(t) \neq q_i(t), \forall t \in [0, T], i = 1, 2, 3 \end{array} \right\}.$$

Our main results are the following:

**Theorem 1.1** Let  $T = 1$ , for the values of  $m_1, m_2, m_3$  given in Table 1 with  $M = 1$ , the minimizer of  $f(q)$  on the closure  $\overline{\Lambda}_-$  of  $\Lambda_-$  is a non-collision 1-periodic solution of (1.6); for the values of  $m_1, m_2, m_3$  given in Table 2 with  $m_1 = m_2 = m_3 = 1$ , the minimizer of  $f(q)$  on  $\overline{\Lambda}_-$  is a non-collision 1-periodic solution of (1.6).

**Remark 1** In proving Theorem 1, we need to use test functions. We find that if the test functions are circular orbits, we can not get the desired results on  $\overline{\Lambda}_-$ . Therefore, we select elliptic orbits as test functions.

**Theorem 1.2** Let  $T = 1$ , for the values of  $m_1, m_2, m_3$  given in Table 3 with  $M = 1$ , the minimizer of  $f(q)$  on the closure  $\overline{\Lambda}_+$  of  $\Lambda_+$  is a non-collision 1-periodic solution of (1.6); for the values of  $m_1, m_2, m_3$  given in Table 4 with  $m_1 = m_2 = m_3 = 1$ , the minimizer of  $f(q)$  on  $\overline{\Lambda}_+$  is a non-collision 1-periodic solution of (1.6).

**Remark 2** When we take elliptic orbits as test functions, we find that the biggest symmetric space is the anti-T/2 symmetric loop space if the wingding number  $n$  is odd ( $n = \pm 1, \pm 3, \dots$ ); we can not find suitable symmetric space if the wingding number is even. When the wingding number  $n \neq \pm 1$  and we take circular orbits as test functions, we find that the biggest symmetric space is

$$\Lambda = \left\{ q \in W^{1,2}(R/TZ, R^2) \left| \begin{array}{l} q(t + \frac{T}{|n-1|}) = R(|n-1|)q(t), \deg(q - q_1) = n, \\ q(t) \neq q_i(t), \forall t \in [0, T], i = 1, 2, 3 \end{array} \right. \right\},$$

where

$$R(|n-1|) = \begin{pmatrix} \cos \frac{2\pi}{|n-1|} & -\sin \frac{2\pi}{|n-1|} \\ \sin \frac{2\pi}{|n-1|} & \cos \frac{2\pi}{|n-1|} \end{pmatrix} \in SO(2)$$

is a counter-clockwise rotation of angle  $\frac{2\pi}{|n-1|}$  in  $R^2$ . But the Lagrangian actions on the circular test orbits are bigger than the lower bound for the Lagrangian actions on collision symmetric orbits. Hence we consider the anti-T/2 symmetric loop spaces  $\Lambda_{\pm}$ .

## 2 Preliminaries

In this section, we will list some basic Lemmas and inequality for proving our Theorems 1.1 and 1.2.

**Lemma 2.1**(Tonelli[1],[11]) Let  $X$  be a reflexive Banach space,  $S$  be a weakly closed subset of  $X$ ,  $f : S \rightarrow R \cup \{+\infty\}$ . If  $f \not\equiv +\infty$  is weakly lower semi-continuous and coercive ( $f(x) \rightarrow +\infty$  as  $\|x\| \rightarrow +\infty$ ), then  $f$  attains its infimum on  $S$ .

**Lemma 2.2**(Poincare-Wirtinger Inequality[10]) Let  $q \in W^{1,2}(R/TZ, R^K)$  and  $\int_0^T q(t)dt = 0$ , then

$$\int_0^T |q(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |\dot{q}(t)|^2 dt.$$

**Lemma 2.3**(Palais's Symmetry Principle([12])) Let  $\sigma$  be an orthogonal representation of a finite or compact group  $G$ ,  $H$  be a real Hilbert space,  $f : H \rightarrow R$  satisfies  $f(\sigma \cdot x) = f(x), \forall \sigma \in G, \forall x \in H$ .

Set  $F = \{x \in H | \sigma \cdot x = x, \forall \sigma \in G\}$ . Then the critical point of  $f$  in  $F$  is also a critical point of  $f$  in  $H$ .

**Remark 2.1** By Palais's Symmetry Principle and the perturbation invariance for wingding numbers, we know that the critical point of  $f(q)$  in  $\Lambda_{\pm}$  is a periodic solution of Newtonian equation (1.6).

**Lemma 2.4**

(1)(Gordon's Theorem[17]) Let  $x \in W^{1,2}([t_1, t_2], R^K)$  and  $x(t_1) = x(t_2) = 0$ . Then for any  $a > 0$ , we have

$$\int_{t_1}^{t_2} \left( \frac{1}{2} |\dot{x}|^2 + \frac{a}{|x|} \right) dt \geq \frac{3}{2} (2\pi)^{2/3} a^{2/3} (t_2 - t_1)^{1/3}.$$

(2)(Long-Zhang[18]) Let  $x \in W^{1,2}(R/TZ, R^K)$ ,  $\int_0^T x dt = 0$ , then for any  $a > 0$ , we have

$$\int_0^T \left( \frac{1}{2} |\dot{x}|^2 + \frac{a}{|x|} \right) dt \geq \frac{3}{2} (2\pi)^{2/3} a^{2/3} T^{1/3}.$$

**Lemma 2.5** Let  $M = m_1 + m_2 + m_3$ , we have  $l = \sqrt[3]{\frac{MT^2}{4\pi^2}}$ .

**Proof.** It follows from (1.1) and (1.2) that

$$\ddot{q}_1 = m_2 \frac{q_2 - q_1}{|q_2 - q_1|^3} + m_3 \frac{q_3 - q_1}{|q_3 - q_1|^3}. \quad (2.1)$$

Then by (1.3)-(1.5), we obtain

$$\begin{aligned} -\frac{4\pi^2}{T^2} q_1 &= \frac{1}{l^3} (m_2 q_2 + m_3 q_3 - m_2 q_1 - m_3 q_1) \\ &= \frac{1}{l^3} (-m_1 q_1 - m_2 q_1 - m_3 q_1), \end{aligned} \quad (2.2)$$

which implies

$$l^3 = \frac{MT^2}{4\pi^2}, \quad (2.3)$$

that is,

$$l = \sqrt[3]{\frac{MT^2}{4\pi^2}}. \quad \square \quad (2.4)$$

**Lemma 2.6** The radius  $r_1, r_2, r_3$  of the planar circular orbits for the masses  $m_1, m_2, m_3$  are

$$\begin{aligned} r_1 &= \frac{\sqrt{m_2^2 + m_2 m_3 + m_3^2}}{M} l, \\ r_2 &= \frac{\sqrt{m_1^2 + m_1 m_3 + m_3^2}}{M} l, \\ r_3 &= \frac{\sqrt{m_1^2 + m_1 m_2 + m_2^2}}{M} l. \end{aligned}$$

**Proof.** Choose the geometrical center of the initial configuration  $(q_1(0), q_2(0), q_3(0))$  as the origin of the coordinate (x,y). Without loss of generality, by (1.5), we suppose the location coordinates of  $q_1(0), q_2(0), q_3(0)$  are  $A_1(\frac{\sqrt{3}l}{3}, 0), A_2(-\frac{\sqrt{3}l}{6}, \frac{l}{2}), A_3(-\frac{\sqrt{3}l}{6}, -\frac{l}{2})$ . Then we can get the coordinate of the center of masses  $m_1, m_2, m_3$  is  $C(\frac{\frac{\sqrt{3}}{3}m_1l - \frac{\sqrt{3}}{6}m_2l - \frac{\sqrt{3}}{6}m_3l}{M}, \frac{\frac{m_2}{2}l - \frac{m_3}{2}l}{M})$ . To make sure the Assumption (1.4) holds, we introduce the new coordinate

$$\begin{cases} X = x - \frac{\frac{\sqrt{3}}{3}m_1l - \frac{\sqrt{3}}{6}m_2l - \frac{\sqrt{3}}{6}m_3l}{M}, \\ Y = y - \frac{\frac{m_2}{2}l - \frac{m_3}{2}l}{M}. \end{cases}$$

Hence in the new coordinate (X,Y), the location coordinates of  $q_1(0), q_2(0), q_3(0)$  are  $A_1(\frac{\frac{\sqrt{3}}{2}m_2l + \frac{\sqrt{3}}{2}m_3l}{M}, \frac{-\frac{m_2}{2}l + \frac{m_3}{2}l}{M})$ ,  $A_2(-\frac{\frac{\sqrt{3}}{2}m_1l}{M}, \frac{\frac{m_1}{2}l + m_3l}{M})$ ,  $A_3(-\frac{\frac{\sqrt{3}}{2}m_1l}{M}, -\frac{\frac{m_1}{2}l + m_2l}{M})$  and the center of masses  $m_1, m_2, m_3$  is at the origin  $O(0,0)$ . Then compared with (1.3), we have

$$r_1 = |A_1O| = \frac{\sqrt{m_2^2 + m_2m_3 + m_3^2}}{M}l, \quad (2.5)$$

$$r_2 = |A_2O| = \frac{\sqrt{m_1^2 + m_1m_3 + m_3^2}}{M}l, \quad (2.6)$$

$$r_3 = |A_3O| = \frac{\sqrt{m_1^2 + m_1m_2 + m_2^2}}{M}l, \quad (2.7)$$

and

$$\sin \theta_1 = \frac{-m_2 + m_3}{2\sqrt{m_2^2 + m_2m_3 + m_3^2}}, \quad \cos \theta_1 = \frac{\sqrt{3}(m_2 + m_3)}{2\sqrt{m_2^2 + m_2m_3 + m_3^2}}, \quad (2.8)$$

$$\sin \theta_2 = \frac{m_1 + 2m_3}{2\sqrt{m_1^2 + m_1m_3 + m_3^2}}, \quad \cos \theta_2 = -\frac{\sqrt{3}m_1}{2\sqrt{m_1^2 + m_1m_3 + m_3^2}}, \quad (2.9)$$

$$\sin \theta_3 = -\frac{m_1 + 2m_2}{2\sqrt{m_1^2 + m_1m_2 + m_2^2}}, \quad \cos \theta_3 = -\frac{\sqrt{3}m_1}{2\sqrt{m_1^2 + m_1m_2 + m_2^2}}. \quad \square \quad (2.10)$$

### 3 Proof of Theorems

In order to get Theorems, we need two steps to complete the proof.

Step 1: We will establish the existence of variational minimizers of  $f(q)$  in (1.8) on  $\bar{\Lambda}_\pm$ .

**Lemma 3.1**  $f(q)$  in (1.8) attains its infimum on  $\bar{\Lambda}_\pm$ .

**Proof.** By using Lemma 2.2, for  $\forall q \in \Lambda_\pm$ , we can get that the equivalent norm of (1.7) in  $\bar{\Lambda}_\pm$  is

$$\|q\| \cong \left[ \int_0^T |\dot{q}|^2 dt \right]^{\frac{1}{2}}. \quad (3.1)$$

Hence by the definition of  $f(q)$ ,  $f$  is coercive on  $\bar{\Lambda}_\pm$ . Next, we claim that  $f$  is weakly lower semi-continuous on  $\bar{\Lambda}_\pm$ . In fact, for  $\forall q^k \in \Lambda_\pm$ , if  $q^k \rightharpoonup q$  weakly, by compact embedding theorem, we have the uniformly convergence:

$$\max_{0 \leq t \leq T} |q^k(t) - q(t)| \rightarrow 0, \quad k \rightarrow \infty, \quad (3.2)$$

which implies

$$\int_0^T \sum_{i=1}^3 \frac{m_i}{|q^k - q_i|} dt \rightarrow \int_0^T \sum_{i=1}^3 \frac{m_i}{|q - q_i|} dt. \quad (3.3)$$

It is well-known that the norm and its square are weakly lower semi-continuous. Therefore, combined with (3.3), we obtain

$$\liminf_{k \rightarrow \infty} f(q^k) \geq f(q), \quad (3.4)$$

that is,  $f$  is weakly lower semi-continuous on  $\bar{\Lambda}_\pm$ . By Lemma 2.1, we can get that  $f(q)$  in (1.8) attains its infimum on  $\bar{\Lambda}_\pm$ .  $\square$

Step 2: We will prove the variational minimizers in Lemma 3.1 is the noncollision T-period solution of (1.6).

For any collision generalized solution  $q$ , we can estimate the lower bound for the value of Lagrangian action functional.

**Lemma 3.2** For  $\partial\Lambda_{\pm} = \{q \in W^{1,2}(R/TZ, R^2) | q(t + \frac{T}{2}) = -q(t), \exists 1 \leq i_0^{\pm} \leq 3, t_{i_0^{\pm}} \in [0, T] \text{ s.t. } q_{i_0^{\pm}}(t_{i_0^{\pm}}) = q(t_{i_0^{\pm}})\}$ , we have

$$\inf_{q \in \partial\Lambda_{\pm}} f(q) \geq \frac{3}{2}(2\pi)^{2/3}CM^{-1/3}T^{1/3} \triangleq d_1,$$

where

$$C = \min \left\{ \begin{array}{l} 2^{\frac{2}{3}}m_1 + m_2 + m_3 - \frac{1}{3M}(m_1m_2 + m_1m_3 + m_2m_3), \\ 2^{\frac{2}{3}}m_2 + m_1 + m_3 - \frac{1}{3M}(m_1m_2 + m_1m_3 + m_2m_3), \\ 2^{\frac{2}{3}}m_3 + m_1 + m_2 - \frac{1}{3M}(m_1m_2 + m_1m_3 + m_2m_3) \end{array} \right\}.$$

**Proof.** It follows from (1.4) that

$$\sum_{i=1}^3 m_i \dot{q}_i = 0, \quad (3.5)$$

which implies

$$\begin{aligned} \sum_{i=1}^3 m_i |\dot{q} - \dot{q}_i|^2 &= \sum_{i=1}^3 m_i (|\dot{q}|^2 + |\dot{q}_i|^2 - 2\langle \dot{q}, \dot{q}_i \rangle) \\ &= M|\dot{q}|^2 + \sum_{i=1}^3 m_i |\dot{q}_i|^2 - 2\left\langle \dot{q}, \sum_{i=1}^3 m_i \dot{q}_i \right\rangle \\ &= M|\dot{q}|^2 + \sum_{i=1}^3 m_i |\dot{q}_i|^2. \end{aligned} \quad (3.6)$$

Therefore

$$|\dot{q}|^2 = \frac{1}{M} \sum_{i=1}^3 m_i (|\dot{q} - \dot{q}_i|^2 - |\dot{q}_i|^2). \quad (3.7)$$

Hence

$$\begin{aligned} f(q) &= \int_0^T \left[ \frac{1}{2}|\dot{q}|^2 + \sum_{i=1}^3 \frac{m_i}{|q - q_i|} \right] dt \\ &= \frac{1}{M} \int_0^T \sum_{i=1}^3 m_i \left[ \frac{1}{2}|\dot{q} - \dot{q}_i|^2 + \frac{M}{|q - q_i|} \right] dt - \frac{1}{2M} \int_0^T \sum_{i=1}^3 m_i |\dot{q}_i|^2 dt. \end{aligned} \quad (3.8)$$

If  $q \in \bar{\Lambda}_-$  is a collision generalized solution, then there exists  $t_{i_0^-} \in [0, T]$  and  $1 \leq i_0^- \leq 3$  such that  $q(t_{i_0^-}) = q_{i_0^-}(t_{i_0^-})$ . Since  $q_i(t + \frac{T}{2}) = -q_i(t)$ , we obtain  $q(t_{i_0^-} + \frac{kT}{2}) = q_{i_0^-}(t_{i_0^-} + \frac{kT}{2})$ ,  $\forall 0 \leq k \leq 2$ . So, by (1) of Lemma 2.4, we get

$$\begin{aligned} \frac{1}{M} \int_0^T m_{i_0^-} \left[ \frac{1}{2}|\dot{q} - \dot{q}_{i_0^-}|^2 + \frac{M}{|q - q_{i_0^-}|} \right] dt &= \frac{2}{M} m_{i_0^-} \int_0^{\frac{T}{2}} \left[ \frac{1}{2}|\dot{q} - \dot{q}_{i_0^-}|^2 + \frac{M}{|q - q_{i_0^-}|} \right] dt \\ &\geq \frac{3}{2}(2\pi)^{2/3}2^{2/3}m_{i_0^-}M^{-1/3}T^{1/3}. \end{aligned} \quad (3.9)$$

For noncollision pair  $q, q_i (i \neq i_0^-)$ , we have  $\int_0^T q(t)dt = 0$ ,  $\int_0^T q_i(t)dt = 0$ . Therefore  $\int_0^T (q(t) - q_i(t))dt = 0$ . Hence by (2) of Lemma 2.4, we can get

$$\frac{1}{M} \int_0^T \sum_{i \neq i_0^-} m_i \left[ \frac{1}{2}|\dot{q} - \dot{q}_i|^2 + \frac{M}{|q - q_i|} \right] dt \geq \frac{3}{2}(2\pi)^{2/3}(M - m_{i_0^-})M^{-1/3}T^{1/3}. \quad (3.10)$$

For the other term of  $f$ , using the expression for the orbits  $q_1, q_2, q_3$  as in (1.3), Lemma 2.5 and Lemma 2.6, we obtain

$$-\frac{1}{2M} \int_0^T \sum_{i=1}^3 m_i |\dot{q}_i|^2 dt = -\frac{1}{2} (2\pi)^{2/3} (m_1 m_2 + m_1 m_3 + m_2 m_3) M^{-4/3} T^{1/3}. \quad (3.11)$$

Therefore, it follows from (3.9) - (3.11) that

$$\inf_{q \in \partial \Lambda_-} f(q) \geq \frac{3}{2} (2\pi)^{2/3} C M^{-1/3} T^{1/3} \triangleq d_1, \quad (3.12)$$

where

$$C = \min \left\{ \begin{array}{l} 2^{\frac{2}{3}} m_1 + m_2 + m_3 - \frac{1}{3M} (m_1 m_2 + m_1 m_3 + m_2 m_3), \\ 2^{\frac{2}{3}} m_2 + m_1 + m_3 - \frac{1}{3M} (m_1 m_2 + m_1 m_3 + m_2 m_3), \\ 2^{\frac{2}{3}} m_3 + m_1 + m_2 - \frac{1}{3M} (m_1 m_2 + m_1 m_3 + m_2 m_3) \end{array} \right\}.$$

Similarly, if  $q \in \bar{\Lambda}_+$  is a collision generalized solution, we have

$$\inf_{q \in \partial \Lambda_+} f(q) \geq \frac{3}{2} (2\pi)^{2/3} C M^{-1/3} T^{1/3} \triangleq d_1, \quad \square \quad (3.13)$$

**Proof of Theorem 1.1** In order to get Theorem 1.1, we are going to find a test loop  $\tilde{q} \in \Lambda_-$  such that  $f(\tilde{q}) \leq d_2$ . Then the minimizer of  $f$  on  $\bar{\Lambda}_-$  must be a noncollision solution if  $d_2 < d_1$ .

Let  $a > 0, b > 0, \theta \in [0, 2\pi)$  and

$$\tilde{q} - q_1 = \left( a \cos \left( -\frac{2\pi}{T}t + \theta \right), b \sin \left( -\frac{2\pi}{T}t + \theta \right) \right)^T. \quad (3.14)$$

Hence

$$\begin{aligned} \tilde{q} - q_2 &= \tilde{q} - q_1 + q_1 - q_2 \\ &= (q_1 - q_2) + (\tilde{q} - q_1) \\ &= \left( r_1 \cos \left( \frac{2\pi}{T}t + \theta_1 \right) - r_2 \cos \left( \frac{2\pi}{T}t + \theta_2 \right) + a \cos \left( -\frac{2\pi}{T}t + \theta \right), r_1 \sin \left( \frac{2\pi}{T}t + \theta_1 \right) \right. \\ &\quad \left. - r_2 \sin \left( \frac{2\pi}{T}t + \theta_2 \right) + b \sin \left( -\frac{2\pi}{T}t + \theta \right) \right)^T, \end{aligned} \quad (3.15)$$

$$\begin{aligned} \tilde{q} - q_3 &= \left( r_1 \cos \left( \frac{2\pi}{T}t + \theta_1 \right) - r_3 \cos \left( \frac{2\pi}{T}t + \theta_3 \right) + a \cos \left( -\frac{2\pi}{T}t + \theta \right), r_1 \sin \left( \frac{2\pi}{T}t + \theta_1 \right) \right. \\ &\quad \left. - r_3 \sin \left( \frac{2\pi}{T}t + \theta_3 \right) + b \sin \left( -\frac{2\pi}{T}t + \theta \right) \right)^T. \end{aligned} \quad (3.16)$$

It is easy to see that  $\tilde{q} \in \Lambda_-$  and

$$|\dot{\tilde{q}} - \dot{q}_1|^2 = \left( \frac{2\pi}{T} \right)^2 \left[ \frac{a^2 + b^2}{2} - \frac{a^2 - b^2}{2} \cos \left( \frac{4\pi}{T}t - 2\theta \right) \right], \quad (3.17)$$

$$|\tilde{q} - q_1| = \sqrt{\frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos \left( \frac{4\pi}{T}t - 2\theta \right)}, \quad (3.18)$$

$$\begin{aligned}
|\dot{\tilde{q}} - \dot{q}_2|^2 &= \left(\frac{2\pi}{T}\right)^2 \left\{ \frac{a^2 + b^2}{2} - \frac{a^2 - b^2}{2} \cos\left(\frac{4\pi}{T}t - 2\theta\right) + r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1) \right. \\
&\quad - (a+b) \left[ r_1 \cos\left(\frac{4\pi}{T}t + \theta_1 - \theta\right) - r_2 \cos\left(\frac{4\pi}{T}t + \theta_2 - \theta\right) \right] \\
&\quad \left. + (a-b) [r_1 \cos(\theta_1 + \theta) - r_2 \cos(\theta_2 + \theta)] \right\}, \tag{3.19}
\end{aligned}$$

$$\begin{aligned}
|\tilde{q} - q_2| &= \left\{ \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos\left(\frac{4\pi}{T}t - 2\theta\right) + r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1) \right. \\
&\quad + (a+b) \left[ r_1 \cos\left(\frac{4\pi}{T}t + \theta_1 - \theta\right) - r_2 \cos\left(\frac{4\pi}{T}t + \theta_2 - \theta\right) \right] \\
&\quad \left. + (a-b) [r_1 \cos(\theta_1 + \theta) - r_2 \cos(\theta_2 + \theta)] \right\}^{\frac{1}{2}}, \tag{3.20}
\end{aligned}$$

$$\begin{aligned}
|\dot{\tilde{q}} - \dot{q}_3|^2 &= \left(\frac{2\pi}{T}\right)^2 \left\{ \frac{a^2 + b^2}{2} - \frac{a^2 - b^2}{2} \cos\left(\frac{4\pi}{T}t - 2\theta\right) + r_1^2 + r_3^2 - 2r_1r_3\cos(\theta_3 - \theta_1) \right. \\
&\quad - (a+b) \left[ r_1 \cos\left(\frac{4\pi}{T}t + \theta_1 - \theta\right) - r_3 \cos\left(\frac{4\pi}{T}t + \theta_3 - \theta\right) \right] \\
&\quad \left. + (a-b) [r_1 \cos(\theta_1 + \theta) - r_3 \cos(\theta_3 + \theta)] \right\}, \tag{3.21}
\end{aligned}$$

$$\begin{aligned}
|\tilde{q} - q_3| &= \left\{ \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos\left(\frac{4\pi}{T}t - 2\theta\right) + r_1^2 + r_3^2 - 2r_1r_3\cos(\theta_3 - \theta_1) \right. \\
&\quad + (a+b) \left[ r_1 \cos\left(\frac{4\pi}{T}t + \theta_1 - \theta\right) - r_3 \cos\left(\frac{4\pi}{T}t + \theta_3 - \theta\right) \right] \\
&\quad \left. + (a-b) [r_1 \cos(\theta_1 + \theta) - r_3 \cos(\theta_3 + \theta)] \right\}^{\frac{1}{2}}, \tag{3.22}
\end{aligned}$$

$$|\dot{q}_1|^2 = \left(\frac{2\pi}{T}\right)^2 r_1^2, \quad |\dot{q}_2|^2 = \left(\frac{2\pi}{T}\right)^2 r_2^2, \quad |\dot{q}_3|^2 = \left(\frac{2\pi}{T}\right)^2 r_3^2. \tag{3.23}$$

Therefore by (3.17)-(3.23), we get

$$\begin{aligned}
f(\tilde{q}) &= \frac{1}{M} \int_0^T \sum_{i=1}^3 m_i \left[ \frac{1}{2} |\dot{\tilde{q}} - \dot{q}_i|^2 + \frac{M}{|\tilde{q} - q_i|} \right] dt - \frac{1}{2M} \int_0^T \sum_{i=1}^3 m_i |\dot{q}_i|^2 dt \\
&= \frac{2\pi^2}{T} \left\{ \frac{a^2 + b^2}{2} + \frac{m_2 + m_3 - m_1}{M} r_1^2 - \frac{2m_2r_2\cos(\theta_2 - \theta_1) + 2m_3r_3\cos(\theta_3 - \theta_1)}{M} r_1 \right. \\
&\quad + \frac{m_2(a-b)}{M} [r_1 \cos(\theta_1 + \theta) - r_2 \cos(\theta_2 + \theta)] + \frac{m_3(a-b)}{M} [r_1 \cos(\theta_1 + \theta) \\
&\quad \left. - r_3 \cos(\theta_3 + \theta)] \right\} + m_1 \int_0^T \left[ \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos\left(\frac{4\pi}{T}t - 2\theta\right) \right]^{-\frac{1}{2}} dt \\
&\quad + \sum_{i=2}^3 \int_0^T m_i \left\{ \frac{a^2 + b^2}{2} + \frac{a^2 - b^2}{2} \cos\left(\frac{4\pi}{T}t - 2\theta\right) + r_1^2 + r_i^2 - 2r_1r_i\cos(\theta_i - \theta_1) \right. \\
&\quad + (a+b) \left[ r_1 \cos\left(\frac{4\pi}{T}t + \theta_1 - \theta\right) - r_i \cos\left(\frac{4\pi}{T}t + \theta_i - \theta\right) \right] \\
&\quad \left. + (a-b) [r_1 \cos(\theta_1 + \theta) - r_i \cos(\theta_i + \theta)] \right\}^{-\frac{1}{2}} dt \\
&= d_2(a, b, \theta). \tag{3.24}
\end{aligned}$$



In order to estimate  $d_2$ , we have computed the numerical values of  $d_2 = f(q)$  over some selected test loops. The computation of the integral that appears in (3.24) has been done using the function  $\{quad\}$  of Mathematica 7.1 with an error less than  $10^{-6}$ . Let  $T = 1$ , the results of the numerical explorations are given in Table 1 with  $M = 1$  and Table 2 with  $m_1 = m_2 = m_3 = 1$ .

Table 1: Parameters for test loops for Theorem 1.1

a	b	$\theta$	m1	m2	m3	d1	d2
0.13	0.49	$\frac{\pi}{20}$	0.29	0.42	0.29	5.419669	5.417862
0.15	0.49	$\frac{\pi}{20}$	0.29	0.41	0.30	5.417626	5.416591
0.15	0.49	$\frac{\pi}{20}$	0.29	0.42	0.29	5.419669	5.413794
0.15	0.49	$\frac{\pi}{20}$	0.30	0.35	0.35	5.441499	5.436767
0.15	0.51	$\frac{\pi}{20}$	0.30	0.36	0.34	5.441669	5.437985
0.15	0.51	$\frac{\pi}{20}$	0.30	0.37	0.33	5.442180	5.433615
0.15	0.51	$\frac{\pi}{20}$	0.30	0.38	0.32	5.443031	5.429587
0.15	0.51	$\frac{\pi}{20}$	0.30	0.39	0.31	5.444223	5.425898
0.15	0.51	$\frac{\pi}{20}$	0.30	0.40	0.30	5.445755	5.422550
0.15	0.53	$\frac{\pi}{20}$	0.31	0.35	0.34	5.470820	5.467576
0.15	0.53	$\frac{\pi}{20}$	0.31	0.36	0.33	5.471160	5.462971
0.15	0.53	$\frac{\pi}{20}$	0.31	0.37	0.32	5.471841	5.458707
0.15	0.53	$\frac{\pi}{20}$	0.31	0.38	0.31	5.472863	5.454784
0.17	0.45	$\frac{\pi}{20}$	0.32	0.32	0.36	5.500992	5.488608
0.17	0.47	$\frac{\pi}{20}$	0.32	0.33	0.35	5.500481	5.454518
0.17	0.47	$\frac{\pi}{20}$	0.32	0.34	0.34	5.500311	5.449987
0.17	0.47	$\frac{\pi}{20}$	0.33	0.34	0.33	5.530142	5.444254
0.45	0.15	$\pi$	0.33	0.31	0.36	5.471160	5.456006
0.45	0.15	$\pi$	0.33	0.32	0.35	5.500481	5.455325
0.45	0.15	$\pi$	0.33	0.33	0.34	5.530142	5.454984
0.47	0.13	$\pi$	0.34	0.30	0.36	5.441669	5.439671
0.47	0.13	$\pi$	0.34	0.31	0.35	5.470820	5.438820
0.47	0.13	$\pi$	0.34	0.32	0.34	5.500311	5.438309
0.47	0.15	$\pi$	0.35	0.30	0.35	5.441499	5.417900
0.49	0.15	$\pi$	0.36	0.29	0.35	5.412519	5.411552
0.49	0.15	$\pi$	0.36	0.32	0.32	5.500992	5.410020
0.49	0.15	$\pi$	0.37	0.29	0.34	5.412859	5.411962
0.49	0.15	$\pi$	0.37	0.30	0.33	5.442180	5.411281
0.49	0.15	$\pi$	0.37	0.31	0.32	5.471841	5.410940
0.49	0.15	$\pi$	0.38	0.29	0.33	5.413540	5.412712
0.49	0.15	$\pi$	0.38	0.30	0.32	5.443031	5.412201
0.49	0.15	$\pi$	0.38	0.31	0.31	5.472863	5.412031
0.49	0.15	$\pi$	0.39	0.29	0.32	5.414562	5.413803
0.49	0.15	$\pi$	0.39	0.30	0.31	5.444223	5.413462
0.49	0.17	$\pi$	0.40	0.29	0.31	5.415924	5.415807
0.49	0.17	$\pi$	0.40	0.30	0.30	5.445755	5.415637
0.49	0.17	$\pi$	0.41	0.30	0.29	5.417626	5.416078
0.49	0.17	$\pi$	0.42	0.29	0.29	5.419669	5.416689

Table 2: Parameters for test loops for Theorem 1.1

a	b	$\theta$	m1	m2	m3	d1	d2
0.15	0.67	$\frac{\pi}{30}$	1.00	1.00	1.00	11.523843	11.505860
0.15	0.67	$\frac{\pi}{30}$	1.00	1.00	1.00	11.523843	11.505860
0.15	0.69	$\frac{\pi}{30}$	1.00	1.00	1.00	11.523843	11.444212
0.17	0.65	$\frac{\pi}{30}$	1.00	1.00	1.00	11.523843	11.493238
0.17	0.67	$\frac{\pi}{20}$	1.00	1.00	1.00	11.523843	11.452135
0.17	0.69	$\frac{\pi}{20}$	1.00	1.00	1.00	11.523843	11.400124
0.19	0.63	$\frac{\pi}{30}$	1.00	1.00	1.00	11.523843	11.519350
0.19	0.65	$\frac{\pi}{20}$	1.00	1.00	1.00	11.523843	11.455969
0.19	0.67	$\frac{\pi}{20}$	1.00	1.00	1.00	11.523843	11.386608
0.19	0.69	$\frac{\pi}{20}$	1.00	1.00	1.00	11.523843	11.344747
0.61	0.23	$\pi$	1.00	1.00	1.00	11.523843	11.516685
0.63	0.19	$\pi$	1.00	1.00	1.00	11.523843	11.489791
0.63	0.21	$\pi$	1.00	1.00	1.00	11.523843	11.436105
0.65	0.17	$\pi$	1.00	1.00	1.00	11.523843	11.461786
0.65	0.19	$\pi$	1.00	1.00	1.00	11.523843	11.392115
0.65	0.21	$\pi$	1.00	1.00	1.00	11.523843	11.349366
0.67	0.15	$\pi$	1.00	1.00	1.00	11.523843	11.472422
0.67	0.17	$\pi$	1.00	1.00	1.00	11.523843	11.383978
0.67	0.19	$\pi$	1.00	1.00	1.00	11.523843	11.324970
0.67	0.21	$\pi$	1.00	1.00	1.00	11.523843	11.291915
0.69	0.13	$\pi$	1.00	1.00	1.00	11.523843	11.522980
0.69	0.15	$\pi$	1.00	1.00	1.00	11.523843	11.412094
0.69	0.17	$\pi$	1.00	1.00	1.00	11.523843	11.334189
0.69	0.19	$\pi$	1.00	1.00	1.00	11.523843	11.284714

For the parameters  $a, b, \theta$  given in Table 1 and Table 2, we all have  $d_2 < d_1$ . This completes the Proof of Theorem 1.1.  $\square$

**Proof of Theorem 1.2** To get Theorem 1.2, we are going to find a test loop  $\bar{q} \in \Lambda_+$  such that  $f(\bar{q}) \leq d_3$ . Then the minimizer of  $f$  on  $\bar{\Lambda}_+$  must be a noncollision solution if  $d_3 < d_1$ .

Let  $a > 0$ ,  $\theta \in [0, 2\pi)$  and

$$\bar{q} - q_1 = ae^{\sqrt{-1}(\frac{2\pi}{T}t+\theta)}. \quad (3.25)$$

Hence

$$\begin{aligned} \bar{q} - q_2 &= q_1 + ae^{\sqrt{-1}(\frac{2\pi}{T}t+\theta)} - q_2 \\ &= r_1 e^{\sqrt{-1}(\frac{2\pi}{T}t+\theta_1)} - r_2 e^{\sqrt{-1}(\frac{2\pi}{T}t+\theta_2)} + ae^{\sqrt{-1}(\frac{2\pi}{T}t+\theta)}, \end{aligned} \quad (3.26)$$

$$\begin{aligned} \bar{q} - q_3 &= q_1 + ae^{\sqrt{-1}(\frac{2\pi}{T}t+\theta)} - q_3 \\ &= r_1 e^{\sqrt{-1}(\frac{2\pi}{T}t+\theta_1)} - r_3 e^{\sqrt{-1}(\frac{2\pi}{T}t+\theta_3)} + ae^{\sqrt{-1}(\frac{2\pi}{T}t+\theta)}. \end{aligned} \quad (3.27)$$

It is easy to see that  $\bar{q} \in \Lambda_+$  and

$$|\dot{\bar{q}} - \dot{q}_1|^2 = \left(\frac{2\pi}{T}\right)^2 a^2, \quad |\bar{q} - q_1| = a, \quad (3.28)$$

$$|\dot{\bar{q}} - \dot{q}_2|^2 = \left(\frac{2\pi}{T}\right)^2 [a^2 + r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1) + 2ar_1\cos(\theta_1 - \theta) - 2ar_2\cos(\theta_2 - \theta)], \quad (3.29)$$

$$|\bar{q} - q_2| = [a^2 + r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1) + 2ar_1\cos(\theta_1 - \theta) - 2ar_2\cos(\theta_2 - \theta)]^{\frac{1}{2}}, \quad (3.30)$$

$$|\dot{\bar{q}} - \dot{q}_3|^2 = \left(\frac{2\pi}{T}\right)^2 [a^2 + r_1^2 + r_3^2 - 2r_1r_3\cos(\theta_3 - \theta_1) + 2ar_1\cos(\theta_1 - \theta) - 2ar_3\cos(\theta_3 - \theta)], \quad (3.31)$$

$$|\bar{q} - q_3| = [a^2 + r_1^2 + r_3^2 - 2r_1r_3\cos(\theta_3 - \theta_1) + 2ar_1\cos(\theta_1 - \theta) - 2ar_3\cos(\theta_3 - \theta)]^{\frac{1}{2}}, \quad (3.32)$$

$$|\dot{q}_1|^2 = \left(\frac{2\pi}{T}\right)^2 r_1^2, \quad |\dot{q}_2|^2 = \left(\frac{2\pi}{T}\right)^2 r_2^2, \quad |\dot{q}_3|^2 = \left(\frac{2\pi}{T}\right)^2 r_3^2. \quad (3.33)$$

Therefore by (3.28)-(3.33), we get

$$\begin{aligned} f(\bar{q}) &= \frac{1}{M} \int_0^T \sum_{i=1}^3 m_i \left[ \frac{1}{2} |\dot{\bar{q}} - \dot{q}_i|^2 + \frac{M}{|\bar{q} - q_i|} \right] dt - \frac{1}{2M} \int_0^T \sum_{i=1}^3 m_i |\dot{q}_i|^2 dt \\ &= \frac{2\pi^2}{T} \left[ a^2 + \frac{m_2 + m_3 - m_1}{M} r_1^2 - \frac{2m_2r_2\cos(\theta_2 - \theta_1) + 2m_3r_3\cos(\theta_3 - \theta_1)}{M} r_1 \right. \\ &\quad \left. + \frac{2(m_2 + m_3)}{M} ar_1\cos(\theta_1 - \theta) - \frac{2m_2r_2\cos(\theta_2 - \theta) + 2m_3r_3\cos(\theta_3 - \theta)}{M} a \right] \\ &\quad + \frac{m_1T}{a} + m_2 \int_0^T [a^2 + r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1) + 2ar_1\cos(\theta_1 - \theta) \\ &\quad - 2ar_2\cos(\theta_2 - \theta)]^{-\frac{1}{2}} dt + m_3 \int_0^T [a^2 + r_1^2 + r_3^2 - 2r_1r_3\cos(\theta_3 - \theta_1) \\ &\quad + 2ar_1\cos(\theta_1 - \theta) - 2ar_3\cos(\theta_3 - \theta)]^{-\frac{1}{2}} dt \\ &= d_3(a, \theta). \end{aligned} \quad (3.34)$$

In order to estimate  $d_3$ , we have computed the numerical values of  $d_3 = f(q)$  over some selected test loops. The computation of the integral that appears in (3.34) has been done using the function `{quad}` of Mathematica 7.1 with an error less than  $10^{-6}$ . Let  $T = 1$ , the results of the numerical explorations are given in Table 3 with  $M = 1$  and Table 4 with  $m_1 = m_2 = m_3 = 1$ .

Table 3: Parameters for test loops for Theorem 1.2

a	$\theta$	m1	m2	m3	d1	d3
0.17	$\frac{\pi}{2}$	0.10	0.75	0.15	5.062791	5.060773
0.17	$\frac{\pi}{2}$	0.10	0.77	0.13	5.083903	5.071551
0.17	$\frac{\pi}{2}$	0.10	0.78	0.12	5.094969	5.077450
0.17	$\frac{\pi}{2}$	0.10	0.80	0.10	5.118123	5.090270
0.17	$\frac{\pi}{2}$	0.15	0.53	0.32	5.051742	5.050040
0.17	$\frac{\pi}{2}$	0.15	0.57	0.28	5.068768	5.046398
0.17	$\frac{\pi}{2}$	0.15	0.60	0.25	5.085112	5.047242
0.17	$\frac{\pi}{2}$	0.15	0.65	0.20	5.119162	5.055458
0.17	$\frac{\pi}{2}$	0.15	0.70	0.15	5.161725	5.072186
0.17	$\frac{\pi}{2}$	0.15	0.72	0.13	5.121130	5.081261
0.17	$\frac{\pi}{2}$	0.20	0.31	0.49	5.176554	5.175168
0.17	$\frac{\pi}{2}$	0.20	0.35	0.45	5.167020	5.144967
0.17	$\frac{\pi}{2}$	0.20	0.40	0.40	5.162763	5.114876
0.17	$\frac{\pi}{2}$	0.20	0.50	0.30	5.179789	5.080232
0.17	$\frac{\pi}{2}$	0.20	0.55	0.25	5.201070	5.075680
0.17	$\frac{\pi}{2}$	0.20	0.60	0.20	5.230864	5.079639
0.19	$\frac{\pi}{2}$	0.25	0.22	0.53	5.249837	5.237465
0.19	$\frac{\pi}{2}$	0.25	0.25	0.50	5.325541	5.202291
0.19	$\frac{\pi}{2}$	0.25	0.30	0.45	5.308516	5.150479
0.19	$\frac{\pi}{2}$	0.25	0.35	0.40	5.300003	5.107178
0.19	$\frac{\pi}{2}$	0.25	0.62	0.13	5.041112	5.020454
0.19	$\frac{\pi}{2}$	0.30	0.22	0.48	5.230258	5.222385
0.19	$\frac{\pi}{2}$	0.30	0.25	0.45	5.308516	5.189765
0.19	$\frac{\pi}{2}$	0.30	0.30	0.40	5.445755	5.142208
0.19	$\frac{\pi}{2}$	0.30	0.35	0.35	5.441499	5.103164
0.19	$\frac{\pi}{2}$	0.30	0.56	0.14	5.036553	5.032137
0.21	$\frac{\pi}{2}$	0.35	0.21	0.44	5.192935	5.184596
0.21	$\frac{\pi}{2}$	0.35	0.29	0.36	5.412519	5.092092
0.21	$\frac{\pi}{2}$	0.35	0.39	0.26	5.327621	5.007107
0.21	$\frac{\pi}{2}$	0.35	0.48	0.17	5.091316	4.959734
0.21	$\frac{\pi}{2}$	0.35	0.53	0.12	4.971952	4.945333
0.21	$\frac{\pi}{3}$	0.40	0.28	0.32	5.386433	5.342981
0.21	$\frac{\pi}{3}$	0.40	0.32	0.28	5.386433	5.287294
0.21	$\frac{\pi}{3}$	0.40	0.36	0.24	5.271874	5.237055
0.21	$\frac{\pi}{3}$	0.40	0.38	0.22	5.216638	5.213978
0.23	$\frac{\pi}{2}$	0.45	0.19	0.36	5.139742	5.127834
0.23	$\frac{\pi}{2}$	0.45	0.29	0.26	5.337836	5.003006
0.23	$\frac{\pi}{2}$	0.45	0.37	0.18	5.112805	4.927660
0.23	$\frac{\pi}{2}$	0.45	0.46	0.09	4.885693	4.868944
0.23	$\frac{\pi}{2}$	0.50	0.18	0.32	5.123871	5.108878
0.23	$\frac{\pi}{2}$	0.50	0.23	0.27	5.266218	5.044762
0.23	$\frac{\pi}{2}$	0.50	0.29	0.21	5.208258	4.979058
0.23	$\frac{\pi}{2}$	0.50	0.37	0.13	4.990036	4.910522
0.23	$\frac{\pi}{2}$	0.50	0.41	0.09	4.889098	4.884426

Table 4: Parameters for test loops for Theorem 1.2

a	$\theta$	m1	m2	m3	d1	d3
0.21	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	11.327950
0.23	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	11.036769
0.23	$\frac{\pi}{3}$	1.00	1.00	1.00	11.523843	11.336568
0.25	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	10.821272
0.25	$\frac{\pi}{3}$	1.00	1.00	1.00	11.523843	11.187475
0.25	$\frac{\pi}{4}$	1.00	1.00	1.00	11.523843	11.453195
0.27	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	10.667031
0.27	$\frac{\pi}{3}$	1.00	1.00	1.00	11.523843	11.107374
0.27	$\frac{\pi}{4}$	1.00	1.00	1.00	11.523843	11.411685
0.29	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	10.563849
0.29	$\frac{\pi}{3}$	1.00	1.00	1.00	11.523843	11.085761
0.29	$\frac{\pi}{4}$	1.00	1.00	1.00	11.523843	11.430090
0.31	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	10.504424
0.31	$\frac{\pi}{3}$	1.00	1.00	1.00	11.523843	11.114860
0.31	$\frac{\pi}{4}$	1.00	1.00	1.00	11.523843	11.500414
0.33	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	10.483477
0.33	$\frac{\pi}{3}$	1.00	1.00	1.00	11.523843	11.188786
0.35	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	10.497161
0.35	$\frac{\pi}{3}$	1.00	1.00	1.00	11.523843	11.302997
0.37	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	10.542652
0.37	$\frac{\pi}{3}$	1.00	1.00	1.00	11.523843	11.453926
0.39	$\frac{\pi}{2}$	1.00	1.00	1.00	11.523843	10.617860

For the parameters  $a, \theta$  given in Table 3 and Table 4, we all have  $d_3 < d_1$ . This completes the Proof of Theorem 1.2.  $\square$

## References

- [1] A. Ambrosetti, V. Coti Zelati, Periodic Solutions of Singular Lagrangian Systems, Birkhäuser, Basel, 1993.
- [2] A. Bahri, P. Rabinowitz, Periodic solutions of Hamiltonian systems of three body type, Ann. IHP. nonlineaire, 8(6)(1991), 561-649.
- [3] A. Chenciner, R. Montgomery, A remarkable periodic solution of the three body problem in the case of equal masses, Annals of Math., 152(3)(2000), 881-901.
- [4] A. Chenciner, Action minimizing solutions of the Newtonian n-body problem: From homology to symmetry, ICM 2002, Vol. 3, 255-264.
- [5] A. Venturelli, Une caractérisation variationnelle des solutions de Lagrange du probleme plan des trois corps, C. R. Acad. Sci. Paris, 332(7)(2001), 641-644.
- [6] C. Marchal, How the method of minimization of action avoids singularities, Cel. Mech. and Dyn. Astronomy, 83(1-4)(2002), 325-353.
- [7] C. Moore, Braids in classical gravity, Physical Review Letters, 70(24)(1993), 3675-3679.

- [8] C. Simó, New families of solutions in N-body problems, *Progress in Math.*, 201(2001), 101-115.
- [9] D. Ferrario, S. Terracini, On the existence of collisionless equivariant minimizers for the classical n-body problem, *Invention Math.*, 155(2)(2004), 305-362.
- [10] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, 74, Springer-Verlag, New York, 1989.
- [11] M. Struwe, *Variational methods*, Third Edition, Springer-Verlag, 1990.
- [12] R. Palais, The principle of symmetric criticality, *Comm. Math. Phys.*, 69(1)(1979), 19-30.
- [13] S. Q. Zhang, Q. Zhou, A minimizing property of Lagrangian solutions, *Acta Math. Sinica*, 17(3)(2001), 497-500.
- [14] S. Q. Zhang, Q. Zhou, Variational methods for the choreography solution to the three-body problem, *Science in China*, 45(5)(2002), 594-597.
- [15] S. Q. Zhang, Q. Zhou, Nonplanar and noncollision periodic solutions for N-body problems, *DCDS-A*, 10(3)(2004), 679-685.
- [16] U. Bessi, V. Coti Zelati, Symmetries and noncollision closed orbits for planar N-body-type problems, *Nonlinear Anal. TMA*, 16(1991), 587-598.
- [17] W. Gordon, A minimizing property of Keplerian orbits, *Amer. J. Math.*, 99(5)(1977), 961-971.
- [18] Y. M. Long, S. Q. Zhang, Geometric characterizations for variational minimization solutions of the 3-body problems, *Acta Math. Sinica*, 16(4)(2000), 579-592.